

Supplementary File to "Effects of physical erosion on chemical denudation rates: a numerical modeling study of soil-mantled hillslopes"

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In the main text, we presented a summary of the model equations and the assumptions underlying them. Here we present the full derivation of all dimensional and nondimensional equations.

A Derivation of model equations

We begin creating the model by deriving an equation for the rate of change in soil thickness H . We assume that mass is added to the soil only by conversion of bedrock to soil from below. Because we are applying mass conservation to a unit column of soil, the dimensions of soil mass are $[\text{M L}^{-2}]$.

$$\frac{d(\text{soil mass per unit area})}{dt} = \text{mass input rate} - \text{mass loss rate} \quad (\text{A.1})$$

$$\frac{d(\rho_s H)}{dt} = \epsilon_b - D_{inst} \quad (\text{A.2})$$

Here H is the soil thickness [L], ϵ_b is the rate of bedrock lowering due to conversion of bedrock to soil (also termed the soil production rate) [$\text{M L}^{-2} \text{T}^{-1}$], ρ_s is the density of soil [M L^{-3}], and D_{inst} is the denudation rate [$\text{M L}^{-2} \text{T}^{-1}$], defined as the rate of soil mass loss per unit area. We assume that the soil production rate ϵ_b depends exponentially on soil thickness as in the formulation of *Heimsath et al.* (1997, 1999, 2000, 2001, 2005).

$$\epsilon_b = \epsilon_0 e^{-\alpha H} \quad (\text{A.3})$$

Here ϵ_0 represents the soil production rate in the absence of soil, and α [L^{-1}] is a constant that describes the exponential dependence of soil production rate on soil thickness. If we assume ρ_s is constant in time and that D_{inst} is the sum of instantaneous physical erosion rates E_{inst} and instantaneous chemical denudation rates W_{inst} , then the rate of change in soil thickness H is:

$$\frac{dH}{dt} = \frac{1}{\rho_s} (\epsilon_0 e^{-\alpha H} - E_{inst} - W_{inst}). \quad (\text{A.4})$$

Noting that in a well-mixed soil the CRN concentration N_s [atoms M^{-1}] is the total number of CRN atoms in the soil column, n_s [atoms L^{-2}], divided by the mass of the soil column, $\rho_s H$ [M L^{-2}], and assuming that ρ_s is constant in time,

$$\frac{dN_s}{dt} = N_s \left(\frac{1}{n_s} \frac{dn_s}{dt} - \frac{1}{H} \frac{dH}{dt} \right). \quad (\text{A.5})$$

We then assume that the soil gains CRN only through in-situ production of CRN and through incorporation of CRN from bedrock during soil production at the soil-bedrock boundary, and loses CRN only through soil denudation.

$$\begin{aligned} \frac{dn_s}{dt} = & \frac{dn_s}{dt}(\text{in-situ CRN production}) + \frac{dn_s}{dt}(\text{soil production}) \\ & - \frac{dn_s}{dt}(\text{loss to denudation}) \end{aligned} \quad (\text{A.6})$$

Because Equation A.6 neglects radioactive decay, it is suitable for stable CRN (^3He , ^{21}Ne) and CRN with half-lives that are much longer than the residence time of minerals within the penetration depth of cosmogenic radiation (^{10}Be , ^{26}Al), and unsuitable for CRN with much shorter half-lives (e.g., ^{14}C). The production rate of CRN in the soil decreases exponentially with depth, and thus the in-situ production rate of CRN in the soil column is, expressed in [$\text{atoms L}^{-2} \text{T}^{-1}$],

$$\begin{aligned}\frac{dn_s}{dt}(\text{in-situ CRN production}) &= \rho_s \int_0^H P_0 e^{-\rho_s z/\Lambda} dz \\ &= \Lambda P_0 (1 - e^{-\rho_s H/\Lambda}).\end{aligned}\quad (\text{A.7})$$

Defining N_{z_b} as the concentration of CRN in bedrock at the soil-bedrock boundary at depth z_b , the rate of CRN supply to the soil by soil production is N_{z_b} times the soil production rate:

$$\frac{dn_s}{dt}(\text{soil production}) = N_{z_b} \epsilon_0 e^{-\alpha H}. \quad (\text{A.8})$$

N_{z_b} varies in time as the cosmic ray flux produces new CRN at the soil-bedrock boundary (the first term in Equation A.9) and as soil production pushes the soil-bedrock boundary deeper below the surface where CRN concentrations are smaller (the second term).

$$\frac{dN_{z_b}}{dt} = P(z_b) + \frac{\epsilon_0 e^{-\alpha H}}{\rho_r} \left(\frac{\partial N(z \geq z_b)}{\partial z} \right) \quad (\text{A.9})$$

Because the CRN production rate drops off exponentially as it passes through matter, and because CRN concentrations below the soil-bedrock boundary drop off exponentially according to Equation A.10,

$$N(z \geq z_b) = N_{z_b} e^{-\rho_r(z-z_b)/\Lambda}, \quad (\text{A.10})$$

the rate of change of N_{z_b} over time is given by:

$$\frac{dN_{z_b}}{dt} = P_0 e^{-\rho_s H/\Lambda} - \epsilon_0 e^{-\alpha H} \frac{N_{z_b}}{\Lambda}. \quad (\text{A.11})$$

The loss of CRN through soil denudation is the soil CRN concentration N_s times the denudation rate.

$$\begin{aligned}\frac{dn_s}{dt}(\text{loss to denudation}) &= N_s D_{inst} \\ &= N_s (\epsilon_0 e^{-\alpha H} - \rho_s \frac{dH}{dt})\end{aligned}\quad (\text{A.12})$$

The rate of change of n_s is thus

$$\frac{dn_s}{dt} = \Lambda P_0 (1 - e^{-\rho_s H/\Lambda}) + N_{z_b} \epsilon_0 e^{-\alpha H} - N_s (\epsilon_0 e^{-\alpha H} - \rho_s \frac{dH}{dt}), \quad (\text{A.13})$$

which can be substituted into Equation A.5 to yield an expression for CRN concentrations in a well-mixed soil.

$$\frac{dN_s}{dt} = \frac{1}{\rho_s H} \left(\epsilon_0 e^{-\alpha H} (N_{z_b} - N_s) + \Lambda P_0 (1 - e^{-\rho_s H / \Lambda}) \right) \quad (\text{A.14})$$

Lastly, we derive an expression for the rate of change of mineral concentrations in the soil column. We define the average concentration of mineral X in soil, $[X]_s$ [mol M⁻¹], as the number of moles of mineral X in the soil column, Q_X [mol L⁻²], divided by the mass of the soil column, $\rho_s H$ [M L⁻²]. Thus the rate of change of $[X]_s$ is, assuming a constant soil density,

$$\frac{d[X]_s}{dt} = \frac{1}{\rho_s H} \left(\frac{dQ_X}{dt} - [X]_s \rho_s \frac{dH}{dt} \right). \quad (\text{A.15})$$

We now derive an expression for dQ_X/dt to substitute into Equation A.15 by applying conservation of mass to mineral X in the soil column:

$$\begin{aligned} \frac{dQ_X}{dt} = & \frac{dQ_X}{dt} (\text{soil production}) + \frac{dQ_X}{dt} (\text{mineral generation}) \\ & - \frac{dQ_X}{dt} (\text{mineral dissolution}) - \frac{dQ_X}{dt} (\text{physical erosion}). \end{aligned} \quad (\text{A.16})$$

In this expression the first two terms represent inputs of mineral X to the soil, and the last two terms represent losses of mineral X from the soil. The first term is the rate at which mineral X is supplied to the soil by conversion of bedrock to soil, and is given by Equation A.17, in which $[X]_r$ is the concentration of X in bedrock.

$$\frac{dQ_X}{dt} (\text{soil production}) = \epsilon_0 e^{-\alpha H} [X]_r \quad (\text{A.17})$$

This second term is the rate of mineral production within the soil, and we assume it to be nonzero only for secondary minerals. We follow *Chamberlain et al.* (2005) in defining s_X to be the rate of secondary mineral production per unit volume [mol L⁻³ T⁻¹], such that the total rate of secondary mineral production in the soil column is

$$\frac{dQ_X}{dt} (\text{mineral generation}) = s_X H. \quad (\text{A.18})$$

The third term is the rate of mineral dissolution, and we follow the approach of *Chamberlain et al.* (2005) in modeling the dissolution rate of mineral phase

X as a linear function of its specific surface area A_X [$\text{L}^2 \text{mol}^{-1}$] and its concentration in the soil $[X]_s$ [mol M^{-1}].

$$\frac{d[X]_s}{dt}(\text{mineral dissolution}) = -k_X A_X [X]_s \quad (\text{A.19})$$

Here k_X is the dissolution rate constant [$\text{mol L}^{-2} \text{T}^{-1}$]. Over the entire soil column, the total number of moles of mineral X lost to dissolution is then

$$\frac{dQ_X}{dt}(\text{mineral dissolution}) = -k_X A_X [X]_s \rho_s H. \quad (\text{A.20})$$

For all primary minerals we calculate specific surface area A_X as

$$A_X = \frac{6R_X w_X}{\rho_X d_X} \quad (\text{A.21})$$

following *White* (1995), where ρ_X is the density of mineral X [M L^{-3}], w_X is the molar mass of mineral X [M mol^{-1}], d_X is the grain diameter of mineral X [L], and R_X is the surface roughness of mineral X [unitless], which we calculate following *Anbeek et al.* (1994). For secondary minerals, which tend to be much smaller and hence have much larger specific surface areas, we use empirically-determined specific surface areas. Table C.1 provides a list of published values for each of these parameters.

Lastly, the fourth term of the mass balance in Equation A.16 is the number of moles of mineral X lost by physical erosion of soil, and is given by the physical erosion rate times the concentration of mineral X in the soil.

$$\frac{dQ_X}{dt}(\text{physical erosion}) = -E_{inst}[X]_s \quad (\text{A.22})$$

Substitution of Equations A.17, A.18, A.20, and A.22 into Equation A.16 yields the following expression for dQ_X/dt .

$$\frac{dQ_X}{dt} = \epsilon_0 e^{-\alpha H} [X]_r + s_X H - k_X A_X [X]_s \rho_s H - E_{inst}[X]_s \quad (\text{A.23})$$

Finally, substitution of Equations A.23, A.4, and 15 into Equation A.15 yields the rate of change in the concentration of mineral X (Equation A.24). We use Equation A.24 to calculate the evolution of all soil mineral concentrations over time, including the concentration of an insoluble mineral, a quantity that is needed to infer chemical denudation rates via Equation 6.

$$\begin{aligned} \frac{d[X]_s}{dt} = & \frac{\epsilon_0 e^{-\alpha H}}{\rho_s H} ([X]_r - [X]_s) + \frac{s_X}{\rho_s} - k_X A_X [X]_s \\ & + [X]_s \sum_{j=1}^n \left(k_j A_j [X_j]_s w_j - \frac{s_j w_j}{\rho_s} \right) \end{aligned} \quad (\text{A.24})$$

Here n is the number of mineral phases in the soil, $[X_j]_s$ is the concentration of the j^{th} soil mineral phase in the summation [mol M⁻¹], and k_j , A_j , s_j , and w_j are the dissolution rate constant, specific surface area, secondary mineral production rate, and molar mass, respectively, of the j^{th} mineral phase in the summation.

B Derivation of nondimensional model equations

The dimensional differential equations for H , N_s , N_{z_b} , and $[X]_s$ depend on a number of independent parameters. Here we nondimensionalize these equations to remove the model's dependence on soil production parameters ϵ_0 and α and on CRN production parameters P_0 and Λ in order to clarify the dominant controls on soil thickness, CRN concentrations, and soil mineral concentrations. We do not ascribe particular meaning or importance to the nondimensionalizations presented here; we applied these nondimensionalizations merely to eliminate particular dimensional parameters. We note that other nondimensionalization schemes are possible, and that in all schemes nondimensional parameters are easily scaled back to their dimensional counterparts.

We begin nondimensionalizing the model by noting that time can be scaled by a soil production timescale $T_P = \Lambda \epsilon_0^{-1}$, such that nondimensional time \hat{t} is given by

$$\hat{t} = \frac{t}{T_P} = \frac{t \epsilon_0}{\Lambda}. \quad (\text{B.1})$$

Here and below we denote nondimensional quantities with a carat. We nondimensionalize soil density by dividing by $\alpha \Lambda$,

$$\hat{\rho}_s = \frac{\rho_s}{\alpha \Lambda}, \quad (\text{B.2})$$

and scale soil depth with the soil production parameter α .

$$\hat{H} = H \alpha \quad (\text{B.3})$$

We nondimensionalize CRN concentrations N_{z_b} and N_s by scaling them to the hypothetical soil CRN concentration that would occur at a steady-state denudation rate of the maximum soil production rate ϵ_0 .

$$\hat{N}_{z_b} = \frac{N_{z_b}\epsilon_0}{P_0\Lambda}, \hat{N}_s = \frac{N_s\epsilon_0}{P_0\Lambda} \quad (\text{B.4})$$

Molar mineral concentrations are scaled by their molar masses to yield nondimensional concentrations

$$[\hat{X}]_r = [X]_r w_X, [\hat{X}]_s = [X]_s w_X, \quad (\text{B.5})$$

and mineral dissolution and production parameters are scaled as follows.

$$\hat{k}_X = \frac{k_X w_X}{\epsilon_0}, \hat{A}_X = \frac{A_X \Lambda}{w_X}, \hat{s}_X = \frac{s_X w_X \Lambda}{\rho_s \epsilon_0} \quad (\text{B.6})$$

Here \hat{k}_X is the ratio of the dissolution rate of mineral phase X to the maximum soil production rate ϵ_0 , \hat{A}_X is the specific surface area of X relative to the inverse of the gamma-ray neutron penetration depth, and \hat{s}_X relates the clay production rate of X to the maximum soil production rate ϵ_0 . We lastly scale all inferred and instantaneous denudation and erosion rates by the maximum soil production rate ϵ_0 .

$$\begin{aligned} \hat{E}_{inst} &= \frac{E_{inst}}{\epsilon_0} \\ \hat{D}_{inf} &= \frac{D_{inf}}{\epsilon_0} = \frac{1}{\hat{N}_s} \\ \hat{W}_{inf} &= \frac{W_{inf}}{\epsilon_0} = \frac{1}{\hat{N}_s} \left(1 - \frac{[\hat{Z}r]_r}{[\hat{Z}r]_s}\right) \\ \hat{W}_{inst} &= \frac{W_{inst}}{\epsilon_0} = \hat{\rho}_s \hat{H} \sum_{j=1}^n (\hat{k}_j \hat{A}_j [\hat{X}_j]_s - \hat{s}_j) \end{aligned} \quad (\text{B.7})$$

Under the transformations in Equations B.1-B.7, the governing equations assume the following nondimensional forms.

$$\frac{d\hat{H}}{d\hat{t}} = \frac{1}{\hat{\rho}_s} \left(e^{-\hat{H}} - \hat{E}_{inst} - \hat{\rho}_s \hat{H} \sum_{j=1}^n (\hat{k}_j \hat{A}_j [\hat{X}_j]_s - \hat{s}_j) \right) \quad (\text{B.8})$$

$$\frac{d\hat{N}_{z_b}}{d\hat{t}} = e^{-\hat{\rho}_s \hat{H}} - \hat{N}_{z_b} e^{-\hat{H}} \quad (\text{B.9})$$

$$\frac{d\hat{N}_s}{d\hat{t}} = \frac{1}{\hat{\rho}_s \hat{H}} \left(e^{-\hat{H}} (\hat{N}_{zb} - \hat{N}_s) + 1 - e^{-\hat{\rho}_s \hat{H}} \right) \quad (\text{B.10})$$

$$\begin{aligned} \frac{d[\hat{X}]_s}{d\hat{t}} = & \frac{e^{-\hat{H}}}{\hat{\rho}_s \hat{H}} ([\hat{X}]_r - [\hat{X}]_s) + \hat{s}_X - \hat{k}_X \hat{A}_X [\hat{X}]_s + \\ & [\hat{X}]_s \sum_{j=1}^n (\hat{k}_j \hat{A}_j [\hat{X}]_s - \hat{s}_j) \end{aligned} \quad (\text{B.11})$$

C Tables of parameter values used in model runs

Table C.1
Mineral parameter values used in Figure 2

Mineral ^a	Molar		Density	Dissolution		Production		Surface		Relative
	mass	w		rate ^b	k	rate ^c	roughness ^d	area ^e	Solubility ^f	
	($\frac{\text{g}}{\text{mol}}$)	ρ	($\frac{\text{kg}}{\text{m}^3}$)	($\frac{\text{mol}}{\text{m}^2 \text{ yr}}$)	s	($\frac{\text{mol}}{\text{m}^3 \text{ yr}}$)	R	A	$kA/(kA)_{qz}$	
Quartz	60.09	2650	5.0×10^{-8}	0	3	2	1			
Plagioclase	265.44	2650	5.0×10^{-6}	0	39	117	5465			
K-feldspar	278.35	2560	5.0×10^{-7}	0	8	26	122			
Biotite	464.60	3090	2.0×10^{-6}	0	237	1069	19690			
Zircon	183.31	4600	0	0	3	4	0			
Kaolinite	258.18	2600	5.0×10^{-8}	0.51	n/a	4776	2234			

^a We model plagioclase as a standard oligoclase (20% Ca, 80%Na), and assume biotite has the stoichiometric formula $\text{KM}_{0.5}\text{Fe}_{1.5}\text{Si}_3\text{AlO}_{10}(\text{OH})_2$.

^b We use field-based dissolution rate constants for plagioclase, K-feldspar and biotite from *Clow and Drever* (1996), the mean field-based dissolution rate constant from *Schulz and White* (1999) for quartz, and assume zircon is insoluble. In the absence of field-based dissolution rate constants for kaolinite, we assume that the ratio of $k_{kaolinite}$ to $k_{oligoclase}$ in the field is the same as the ratio measured in lab experiments (*Cama et al., 2002; Huertas et al., 1999; White and Brantley, 2003*); that is, we assume that $k_{kaolinite}$ is two orders of magnitude lower than the field-based $k_{oligoclase}$.

^c We assume that the mineral production rate in the soil is zero for each primary mineral, and we use a kaolinite production rate of $0.1 \text{ g kg}^{-1} \text{ yr}^{-1}$, a value that falls within the range of $0.001\text{-}0.2 \text{ g kg}^{-1} \text{ yr}^{-1}$ given in *Barshad* (1957), as cited by *Price et al.* (2005).

^d We calculate surface roughness for each primary mineral following *Anbeek et al.* (1994).

^e For each primary mineral we calculate surface area as $A = 6Rw\rho^{-1}d^{-1}$ following *White* (1995), where d is the mineral grain diameter, which we assume to be 200 microns. Because secondary kaolinite crystals are generally much smaller than 200 microns, we use an experimentally-determined surface area for kaolinite (*Cama et al., 2002*).

^f The relative solubility here is the product of the dissolution rate constant k and the specific surface area A , scaled by the product of k and A for quartz.

Table C.2

Nondimensional parameter values used in Figures 3-7

	$[\hat{X}]_r$	\hat{k}	\hat{A}	\hat{s}
quartz	0.2499	8.0×10^{-9}	3.1×10^4	0
plagioclase	0.4000	3.6×10^{-6}	3.9×10^5	0
K-feldspar	0.2000	3.7×10^{-7}	8.3×10^4	0
biotite	0.1500	2.5×10^{-6}	2.0×10^6	0
zircon	0.0001	0	1.8×10^4	0
kaolinite	0	3.5×10^{-8}	1.6×10^7	0.43
$\hat{\rho}_s = 0.276$				

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