Hillslope process-response models based on the continuity equation

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ABSTRACT. The continuity equation in differential form is the basis for a theoretical analysis of slope development. By including terms for contour curvature in the equation, hillslope and river profiles may be considered as members of a continuous series and analysed together. Slope and river profiles may be deduced from a knowledge of transport processes; and in some cases, process laws can be deduced from measurement of forms. If divide and base level are fixed horizontally, then slope development leads to a 'characteristic form' of hillslopes and river profiles which is independent of the initial form of the profile, and depends solely on the processes of debris transport. This characteristic form may be attained while 50–75 per cent of the initial relief remains. Starting from empirical process studies, characteristic forms are shown to be convex for soil creep, concave for soil wash with gullying, and convexo-concave for ungullied soil wash and for combinations of creep and wash. In a basin where the slope profiles are assumed to have reached characteristic forms, profile measurements can be analysed to yield the transport process laws by which the basin was formed.

THE aim of this paper is to examine a series of process-response models of slope development by combining specific statements about slope processes which are based on field measurement rather than theory, with a general mathematical statement of debris continuity. The merit of this approach to slope studies is that it can act as a springboard for model building in many different directions and with many different limiting conditions, only one example of which can be followed here. In the model discussed below, slopes develop towards a so-called 'characteristic form' which depends solely on the nature and relative rates of the formative processes and not at all on the initial profile of the hillslope, and this development is related to a specific set of boundary conditions which are stated in the model. Mathematical models of slope development are not new, as can be seen, for example, from the pages of A. E. Scheidegger (1961), but only W. E. H. Culling (1965) seems to have fully realized their potential under a set of realistic conditions of removal.

The necessary basis for any process-response model is the continuity equation, which is simply a statement that, if more material is brought into a slope section than is taken out, then the difference must be represented by aggradation within the section; conversely, if less material is brought into a slope section than is removed, the difference must come from net erosion of the section. This statement is true along every line in a drainage basin across which there is no debris transport, that is, along all lines of steepest slope. This includes not only slope profiles as usually understood, but also all drainage lines.

The rate of debris transport is a major term in the continuity equation, and the variation of this term with relief is the critical point at which field studies of process must be applied to produce a realistic model. In considering the development of slope profiles through time, we are necessarily considering a system in 'cyclic time', which S. A. Schumm and R. W. Lichty (1965, p. 113) define as 'a time span encompassing an erosion cycle. . . .

A fluvial system viewed from this perspective is an open system undergoing continuing change . . .'. Therefore our specification of variations in process must be in terms appropriate to cyclic time; that is to say, not in terms of hydraulic variables but in terms of relief variables only, even if the process is hydraulic in nature. The key variables are, therefore, slope gradient and distance from the divide (or its generalization, area drained per unit contour length).

As well as information about the distribution of process along the profile to put in the continuity equation, we need also two other items of information to obtain a solution: (1) the initial form of the slope profiles, and (2) the boundary conditions, namely, the conditions at the divide (usually considered fixed) and the conditions of basal removal at the foot of the slope. The simplest condition of basal removal, and the one which will lead to 'characteristic-form' solutions, is of unimpeded basal removal at a point fixed in both the horizontal and vertical directions, but solutions may be obtained for any specified conditions.

It is now possible to obtain any number of solutions, each a valid sequence of slope forms, but it will be simpler if the choice is limited, for otherwise the mathematics may become intractable. First we will consider only two-dimensional profiles across which there is no transport of debris (that is, lines of greatest slope at right angles to contour lines), and which do not migrate laterally with time. This restricts the solutions obtained to divides, river profiles and profiles in which contours are straight, parallel lines. We will also consider profiles which are developing mainly by mechanical removal rather than by solution. Both of these restrictions have been chosen for the purposes of this paper, and are not implicit in the equations as initially set out below.

Drainage basins, as considered in this type of model, consist of a network of lines of greatest slope, each a line of sediment transport. No distinction need be made between drainage lines and hillslope lines, so that this approach appears to restore to the basin some of the unity expressed in J. Playfair's (1802, p. 114) law of accordant stream junctions and W. M. Davis's graded slopes, but which is absent in treatments which concentrate on the hydraulic system alone, or on the slopes alone. It is the mutual adjustment of rivers and slopes to the transport of material eroded from higher elevations which gives a drainage basin the unity of internal organization which is always one of its most striking features.

THE CONTINUITY EQUATION AND TRANSPORT LAWS

The equations governing the transport of debris down a slope or river profile are as follows (Fig. 1, a and b):

(A) The continuity equation:

$$M+D = -\frac{\partial y}{\partial t} \tag{1}$$

where M is the rate of mechanical lowering,
D is the rate of chemical (dissolved) lowering,
y is the elevation,
and t is time elapsed.

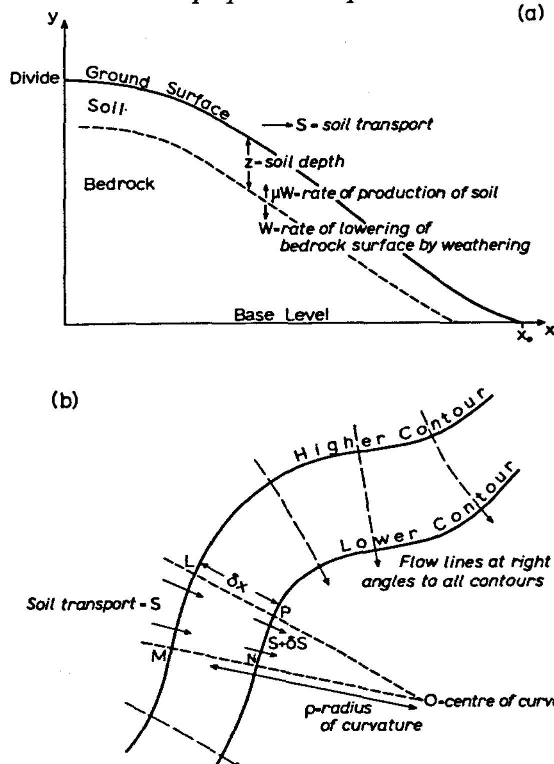

FIGURE 1. (a) Diagrammatic slope profile showing coordinate axes and symbols used in theoretical analysis of slope development; (b) Diagrammatic map showing symbols used in theoretical analysis of slope development where lines of steepest slope converge or diverge (i.e. where contour lines are not straight)

(B) The relationship between mechanical lowering and mechanical transport:

$$M = \nabla \cdot S \tag{2}$$

where ∇ indicates the vector divergence,
and S is the vector sediment transport.
This relationship can be expressed in simpler form if horizontal distance, x, is measured along the line of greatest slope, in which case:

$$M = \frac{\partial S}{\partial x} - \frac{S}{\rho} \tag{3}$$

where ρ is the radius of curvature of the contours, measured as positive where contours are concave outwards, as in hollows.

In considering the continuity equation as two-dimensional, it is implicitly assumed that the lines of greatest slope do not migrate laterally through time. This is a reasonable assumption for spur tops and valley axes, and where contour lines are straight (or on a ridge). Along other lines of greatest slope a three-dimensional solution must be sought. It will also be assumed in the analysis that the contour curvature is constant through time (though varying in space). This is an approximation, even along spur tops and valley axes, which will certainly produce modifications to the forms.

(C) The relationship between rate of lowering and soil thickness:

$$\frac{\partial z}{\partial t} = -\frac{\partial y}{\partial t} - W \tag{4}$$

where z is the soil depth (soil being defined as being material weathered to at least an arbitrary small extent), and W is the rate of lowering of the soil–bedrock interface (defined according to the same criterion). Soil thickness changes by a metre or two while surface and bedrock are lowered through hundreds of metres, so that the rate of change of soil thickness is generally a rather small quantity compared with rate of surface lowering. The most interesting form of this equation, and one which leads to important models of development by solution and of soil development, is for a true dynamic equilibrium in which soil thickness is considered constant. In this case the land surface and the soil–bedrock interface are lowered at the same rate. At the surface, mechanical removal carries away material in its most weathered form, in a state when a proportion, μ, of the original bedrock materials remains undissolved. In this case:

$$M = \mu . W \qquad \text{and } D = (1-\mu) . W$$

Eliminating the rate of bedrock lowering, W, from these two equations, the extent of weathering at the surface, μ, can be calculated as:

$$\mu = \frac{M}{M+D} \tag{5}$$

If, therefore, the soil can be considered in equilibrium, the degree of soil development is related to the relative magnitudes of mechanical and chemical removal. Down-slope, mechanical lowering decreases much more than chemical, so that equilibrium soils become more developed, forming a catena. Models of soil development are therefore implicit in the present discussion of slope development, although this paper is concerned only with slope development by mechanical removal.

(D) The relationship between the actual transport rate, S, and the transporting capacity of the process, C. The two main types of removal condition are:

(i) Transport limited removal: $C = S$ (6)

where C is the transporting capacity of the process (measured in the same units as S).

(ii) Weathering limited removal: $C \gg S$ (7)

Transporting capacity is defined in these equations as the amount that can be carried by the process acting on an unconsolidated soil or other debris deposit.

The two types of transport law, namely transport-limited and weathering-limited removal, were first distinguished by G. K. Gilbert (1877, pp. 96–9). Transport-limited removal occurs where the potential rate of weathering is greater than the rate of transport so that an appreciable soil accumulates, and transport processes operate at their full capacity. Weathering-limited removal occurs where the rate of weathering is lower than the potential capacity of the transporting agents, so that soils are stripped off and slope development is limited by the rate at which rock weathers to loose material. There is perhaps an intermediate condition, applicable to cases where the depth of operation of the transporting process is very variable, for example river bed-load transport. In this case there is some surplus of transporting capacity over volume of unconsolidated material available for removal, and the erosion rate, $-\partial y/\partial t$, is assumed proportional to this surplus. This third condition is defined as 'erosion-limited'.

(iii) Erosion-limited removal: $$-\frac{\partial y}{\partial t} = k(C-S) \tag{8}$$

where k is a constant of the erosion. This additional model is useful in that each of the others can be considered as a special case of it. For, as $k \to \infty$ for an unconsolidated deposit, then necessarily $C = S$, and this is the transport-limited case, (i). Likewise, as $k \to 0$, then $C \gg S$ necessarily if slope development is to proceed at all. The constant k should be related to the ratio

$$G = \frac{\text{Force required to erode a particle from actual deposit}}{\text{Force required to erode same particle from a similar, but unconsolidated deposit}}$$

and the values of k above suggest the following possible relationship:

$$k = \frac{1}{G-1} = \frac{\text{Lifting force against gravity}}{\text{Separating force against cement/cohesion}} \tag{9}$$

(E) The boundary conditions. These will be taken as:

(i) At $x = 0$, $S = 0$ and y is a maximum (10)

This equation states that there is a divide (zero transport) at $x = 0$.

(ii) Also, at $x = x_1$, y is a function of time alone (11)

This states that there is a base level, fixed in horizontal position at distance x_1 from the divide, and which moved vertically in a way which is not controlled by the development of the profile. For simplicity, the time function will commonly be taken to be constantly zero.

(F) The initial form will usually be taken to be a straight slope, for simplicity, although in many cases it will not need to be specified.

(G) The transport law or process law is specified by the transport capacity, C, in terms of morphological factors. Two types of law will be considered. The simpler, which is thought appropriate for most slow mass movements, surface wash and stream transport is

$$C = f(a).\left(-\frac{\partial y}{\partial x}\right)^n \tag{12}$$

where a is the area drained per unit contour length, which is a generalization of distance from the divide in cases of contour curvature; $f(a)$ is a positive function of a, describing the influence of increasing area or distance from the divide; and n is a constant exponent describing the influence of increasing gradient. n is usually considered to be zero or positive.

A more complex law, of which equation (12) is a special case, is

$$\left.\begin{aligned}&\text{for } -\frac{\partial y}{\partial x} \geqslant \tan\alpha\colon C = f(a).\left(-\frac{\partial y}{\partial x} - \tan\alpha\right)^n \\ &\text{for } -\frac{\partial y}{\partial x} \leqslant \tan\alpha\colon C = 0\end{aligned}\right\} \tag{13}$$

where α is a constant angle ($0<\alpha<90°$).
This law is appropriate for processes such as landslides and talus movement, in which the rate of transport increases with gradient only above a critical angle, α. For landslides, such an equation can at best apply only to average rates, and not for the rates of individual earth movements. It may be noted that a high value of the exponent n in equation (12) is able to give a close match to values of transport capacity, C, obtained from equation (13) over a small range of gradients.

The relevance of transport laws of these types may be illustrated by some examples. The simplest transport law may apply to soil creep, which is generally thought to move soil at a rate proportional to the sine of the slope angle (M. J. Kirkby, 1967, pp. 360–3). As a first-order approximation,

$$C \propto \sin\text{(slope angle)} \propto \tan\text{(slope angle)} \propto \left(-\frac{\partial y}{\partial x}\right)$$

If this relationship is compared with equation (12), the distance function, $f(a)$, is a constant, and the exponent $n = 1$. Soil creep therefore conforms approximately to equation (12).

Soil erosion literature contains several formulae of the form $C \propto x^m.\,(\text{slope})^n$, where the constant of proportionality contains soil and precipitation factors (for example, G. W. Musgrave, 1947; A. W. Zingg, 1940; Kirkby, 1969). For rainsplash and movement of surface blocks (Schumm, 1964), the exponents take values of about $m = 0$, $n = 2$. For erosion of fine-grained soil material, the distance exponent rises to $m = 1{\cdot}3$–$1{\cdot}7$, while slope exponents vary within the range $n = 1{\cdot}3$–$2{\cdot}0$.

For rivers, the variation at a station and downstream of hydraulic variables with discharge (L. B. Leopold and T. Maddock, 1953) allows exponents m and n to be calculated on the assumption that the same basic relation applies to both sets of data. Using discharge, q, and sediment load, C, per unit width of flow, then

$$C \propto q^{3\cdot0}(\text{slope})^{2\cdot0}$$

Total discharge is proportional to drainage area with an exponent of 0·75 to 1·0, so that discharge per unit width, q, is proportional to area drained per unit width with an exponent of 0·6 to 1·0
or

$$C \propto a^{2-3}.(\text{slope})^2$$

This calculation for rivers is made on the assumption that rivers flowing over alluvium are always transporting at almost their full capacity, as is shown by the presence of unmoved unconsolidated material in their bed and banks.

Table I summarizes these results, and emphazises the increasing exponents of distance and, to a lesser extent, gradient as the influence of water flow increases from a minimum in soil creep to moderate values for soil wash, and reaches maximum values for river transport.

For a law of transport with a stable slope angle α (equation (13)), scree and rock slopes

TABLE I

Variation of exponents m *and* n *in the empirical relationship* $C \propto a^m . (slope)^n$ *obtained from equation (12)*

Process	*m*	*n*	*Sources*
Soil creep	0	1·0	C. Davison, 1889; Culling, 1963
Rainsplash	0	1–2	Schumm, 1964; Kirkby and Kirkby (unpublished data)
Soil wash	1·3–1·7	1·3–2	Musgrave, 1947, U.S. Agric. Res. Serv., 1961; Zingg, 1940; Kirkby, 1969.
Rivers	2–3	3	Derived from Leopold and Maddock, 1953

provide the simplest example and, for a slope of gradient greater than α, a law of the form

$$C \propto \left(-\frac{\partial y}{\partial x} - \tan \alpha \right)$$

seems appropriate. With an exponent greater than one, the same law roughly applies to shallow landslides in materials with low effective cohesion at failure, for example fissured clays (A. W. Skempton and F. A. Delory, 1957).

These examples show that transport laws of the types contained by equations (12) and (13) are applicable to a wide variety of measured processes, and the form of these equations is able to accommodate an even wider range of possibilities. Transport laws of these types are, therefore, highly relevant to process situations in the field, and will provide a close match of transport law with any real process. Solutions of the continuity equation based on these laws should therefore be able to match many real landforms.

CHARACTERISTIC FORMS

In some solutions of the continuity equation, in the hillslope context, the influence of the initial form of the slope can be shown to decrease rapidly with time, while the slope forms tend closer and closer towards a 'characteristic form', in which the elevation of each point continues to decline with time, but is independent of the initial form. The extent to which erosion must proceed before the actual slope form and the characteristic form become indistinguishable will depend partly on the initial form.

It can be shown for transport equations of the forms shown in equations (12) and (13) in the case where the exponent of slope $n = 1$, and for a fixed base level, $y_1(t) = 0$, that there exists a solution to the continuity equation of the form

$$y = U(x) + V(x) . T(t) \tag{14}$$

where U, V are functions of x alone, and T is a decreasing function of time alone. There is an expression of this form to which the slope profile tends increasingly as time passes, and this form is independent of the initial form of the slope profile. It is reasonable to assume that, for exponents (n) of slope other than 1, a similar form exists to which the slope profile converges, and that approximate solutions to obtain this form will be valid for some finite range of the exponent n. Solutions similar to equation (14) are defined as characteristic forms. As an example of the rate at which a solution can converge towards the characteristic form, Table II and Figure 2 show the development of an initially straight slope by soil

TABLE II

Divide elevations for an initially straight slope on which soil transport is proportional to the tangent of the gradient, as for soil creep. The Table compares exactly computed values of the divide elevation with approximate values obtained as the 'characteristic form' (equation 14).

Relative time taken	0·00	0·01	0·05	0·10	0·20
Exact elevation at divide / Initial elevation of divide	1·000	0·888	0·748	0·643	0·496
Per cent reduction from initial elevation	0	11	25	36	50
Approximate elevation of divide (equation 15) / Initial elevation of divide	0·811	0·790	0·717	0·633	0·495
Per cent error in approximate elevation	19	11	4·1	1·5	0·2

creep. It can be seen that, by the time the initial elevation has been halved, the characteristic form is within 0·2 per cent of the exact solution, so that the two solutions become almost identical while considerable landscape relief is still present.

The restriction that base level is fixed in elevation can be relaxed somewhat without losing the concept of characteristic forms. If the elevation of base level (or of a river at the base of a slope) is not constant, but is proportional to the time function $T(t)$ in equation (14), then the same characteristic form remains a valid solution under particular conditions of a rising or falling base level, if the solutions are considered over a different range of horizontal distance, x. Under conditions of base-level lowering, the range of x will be truncated: for a rising base level, the range of x will be extended.

The solution outlined below refers to the following set of conditions, which have been chosen for their simplicity.

(a) The continuity equation is taken in the form

$$\frac{\partial S}{\partial x} - \frac{S}{\rho} + \mu . \frac{\partial y}{\partial t} = 0 \tag{15}$$

where ρ, the radius of curvature, is considered to vary down the profile, but not to alter with time. This assumption is a reasonable approximation for ridges, for divides and for stream courses.

Chemical weathering is considered to take place in such a way that the extent of weathering at the surface does not vary with position or time; that is, no catena effect can be observed. This assumption is reasonable in areas where chemical weathering is slight.

(b) Removal is considered to be erosion-limited, according to equation (8), where the erosion constant, k, is considered to vary in space but not over time. This mathematical form includes both weathering-limited and transport-limited removal as extreme cases.

(c) The divide is fixed at $x = 0$, and the base-level at $x = x_1, y = 0$.

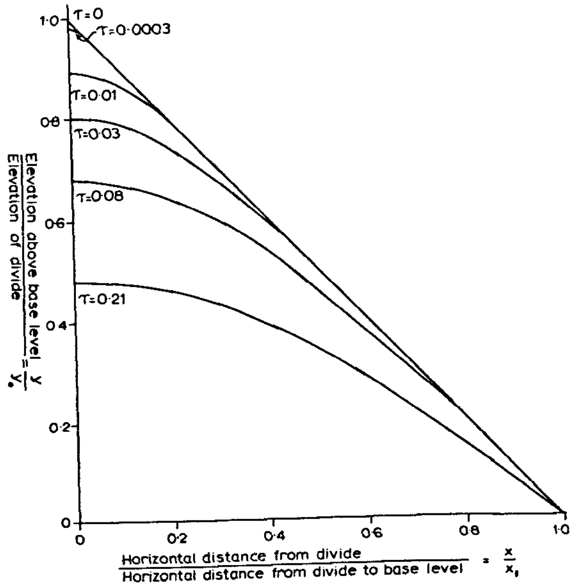

FIGURE 2. Dimensionless graph showing slope development of an initially straight profile under soil creep ($S = C = -\partial y/\partial x$). τ is a dimensionless measure of time elapsed.

(d) The transport law is of the general form shown in equation (13). The area drained per unit contour length, a, in this equation can be shown to be related to the contour radius of curvature, ρ, by the relationship

$$\frac{da}{dx} = 1 + \frac{a}{\rho} \tag{16}$$

Solving this equation for a gives:

$$a = \frac{\int_0^x \Omega \, . \, \partial x}{\Omega}, \text{ where } \Omega = e^{\int_0^x -1/\rho \, \cdot dx} \tag{17}$$

If a solution of the characteristic form (14) exists, then it can be shown that the above set of equations can be reduced to

$$-V'(x) = \left\{ \frac{\lambda \mu \int_0^x \Omega \, . \, V \, .dx + \lambda/k(\Omega \, . \, V)}{\Omega . f(a)} \right\}^{1/n} \tag{18}$$

where λ is a constant chosen to satisfy the boundary conditions; and to:

$$U(x) = (x_1 - x) \tan \alpha, \tag{19}$$

for the range of values over which the elevation is positive, and to $y = 0$ elsewhere.

Applying the inequality $V_0 \geqslant V(x) \geqslant 0$, the solution for $V(x)$ is limited by the inequality:

$$\left\{ V_0^{(n-1)/n} - \frac{n-1}{n} . \lambda^{1/n} . I(x) \right\}^{n/(n-1)} \geqslant V(x) \geqslant V_0 - (\lambda V_0)^{1/n} . I(x) \tag{20}$$

where V_0 is the value of $V(x)$ at $x = 0$, the divide

$$\text{and } I(x) = \int_0^x \left\{\frac{\mu a + 1/k}{f(a)}\right\}^{1/n} . dx \tag{21}$$

The left-hand side of (20) is replaced by the exponential limit at $n = 1$.

The constant, λ, is approximately equal to $\frac{1\cdot 28}{I(x_1)}$ for $n = 1$

and to $\frac{1\cdot 37 V_0}{\{I(x_1)\}^2}$ for $n = 2$, where $I(x_1)$ is the value of $I(x)$ at $x = x_1$, the base-level point. An approximate solution for $V(x)$ is obtained as the mean of the upper bounds of the inequality, and this is satisfactory in many applications.

As an example of the accuracy of this approximation, Figure 3 shows the approximate and the exact solutions for the soil creep case with $S = C = -\partial y/\partial x$, $\mu = 1$, and $\rho \to \infty$ everywhere.

Then $k \to \infty$ and $\Omega = 1$ everywhere, and $I(x) = \int_0^x x\,.dx = \frac{1}{2}x^2$.

The inequality (20) therefore leads to:

$$V_0 e^{-\frac{1}{2}\lambda x^2} \geqslant V(x) \geqslant V_0(1 - \tfrac{1}{2}\lambda x^2),$$

where $\lambda \approx 2\cdot 56/x_1{}^2$. Table III shows the values of $V(x)$ obtained by averaging the upper and lower bounds, and compares it with the exact solution, $V(x) = V_0.\cos \frac{1}{2}\pi(x/x_1)$ (H. S. Carslaw and J. C. Jaeger, 1959).

In a similar way it is possible to obtain an approximate solution for the characteristic form of a slope profile (with the specified boundary conditions of a fixed divide and a fixed base level) for any slope transport law of the forms of equations (12) or (13). Figure 4 shows the approximate solutions for the characteristic form in terms of $I(x)$ for $n = 1$ and 2, and it can be seen that the solution depends much more on $I(x)$ (which itself contains n) than on the exponent n directly. Figure 5 shows the characteristic form profiles obtained for a variety of slope transport laws under the simple conditions that contours are straight ($\rho \to \infty$), there is no chemical solution ($\mu = 1$), and transport is limited ($k \to \infty$). The

TABLE III

Creep example ($S = C = -\partial y/\partial x$), *showing comparison between the exact* $[V(x) = V_0 \cos \frac{1}{2}\pi(x/x_0)]$ *and the approximate solutions* $[V(x) = V_0 e^{-\frac{1}{2}\gamma x^2}$ *and* $V(x) = V_0(1 - \frac{1}{2}\lambda x^2)]$ *for* V(x), *in the characteristic form,* $y = U(x) + V(x).T(t)$, *where* $\lambda = 2\cdot 56/x_1^2$ *and* $U(x) = 0$

$x/x_1 =$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$e^{-\frac{1}{2}\gamma x^2}$	1·000	0·868	0·568	0·278
$1 - \frac{1}{2}\lambda x^2$	1·000	0·858	0·432	−0·280
Mean of above = approximate solutions	1·000	0·863	0·500	−0·001
$\cos \frac{1}{2}\pi(x/x_0)$ = exact solution	1·000	0·866	0·500	0·000

processes listed in Table I are included in this Figure. These profiles seem very promising as predictors of real slope profiles, but it should be remembered that the assumption of straight contours becomes notably false in some cases, particularly in connection with river profiles.

The approximations of equations (20) and (21) seem to provide solutions which are qualitatively reasonable, and it is tempting to aim for even simpler approximations, based on the near-straight lines shown for $I(x)$ in Figure 4. That is, $I(x)$ is approximated by the relation:

$$I(x) = I(x_1).(1 - y/y_0) \tag{22}$$

Substituting into equation (21) for the transport-limited case ($k \to \infty$), the approximate solution is given by:

$$\text{slope} \propto \left\{\frac{a}{f(a)}\right\}^{1/n} \tag{23}$$

or the case of a ridge in which $f(a) \propto a^m$, and $a = x$:

$$y = y_0\left\{1 - \left(\frac{x}{x_1}\right)^{(1-m)/n+1}\right\} \tag{24}$$

The first of these forms is a power law relationship between slope and area, or discharge, which is comparable in form with the results of Leopold and Maddock (1953), or with those of J. T. Hack (1957), if the transport law is modified to allow for differences in bed material. The second of these forms is comparable with the slope profile data collected in the field (Hack and J. C. Goodlett, 1960), and their data can be interpreted as referring to a range of processes with constant values of y_0, x_1; that is to say, there are broad regional similarities of relief and drainage density.

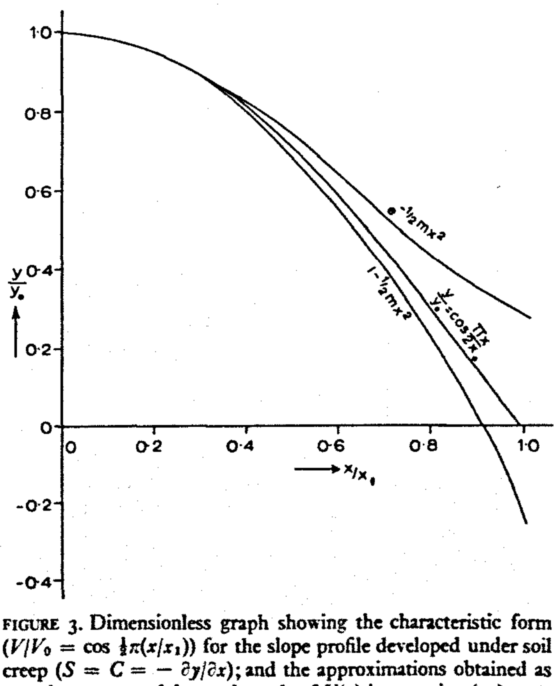

FIGURE 3. Dimensionless graph showing the characteristic form ($V/V_0 = \cos \frac{1}{2}\pi(x/x_1)$) for the slope profile developed under soil creep ($S = C = -\partial y/\partial x$); and the approximations obtained as the upper and lower bounds of $V(x)$ in equation (20)

Without making approximations, some statements can be made about the convexity or concavity of characteristic forms. The importance of this idea is that, if a characteristic form is concave, then it can be stated that profiles developing according to the relevant process law will always tend to become concave, whatever their initial forms. The conditions are obtained by differentiating equation (18), and applying inequalities to the second differential of $V(x)$. In this way sufficient, but not necessary, conditions are obtained as follows:

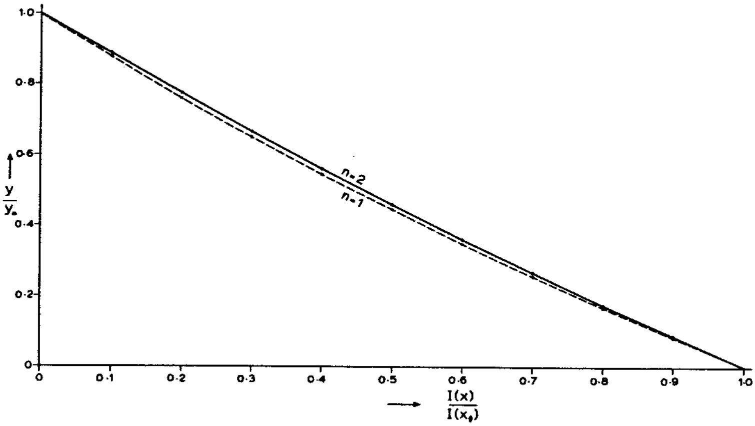

FIGURE 4. Dimensionless graph showing approximate characteristic-form solution for elevation $y/y_0 = V/V_0$, in terms of $I(x)/I(x_1)$, where

$$I(x) = \int_0^x \left\{\frac{\mu a + 1/k}{f(a)}\right\}^{1/n} .dx$$

(equation 21); and for the exponent n taking values 1·0, 2·0. The approximate solutions are obtained as the mean of the upper and lower bounds in equation (20), with λ chosen to satisfy the boundary condition at base level ($V = 0$ at $x = x_1$).

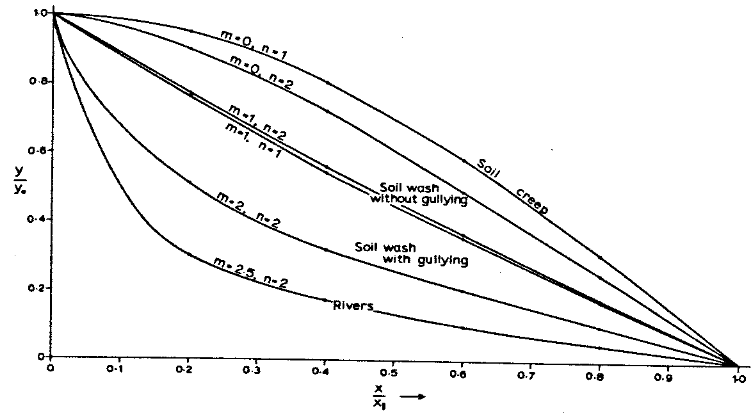

FIGURE 5. Dimensionless graph showing approximate characteristic-form slope profiles for a range of processes from Table I, for the simplest case $\mu = 1$, $k \to \infty$, $\rho \to \infty$. Approximations are on the same basis as in Figure 4.

for concavity:

$$\frac{\dfrac{\partial f(a)}{\partial a}}{f(a)} \geqslant \frac{1}{a + \dfrac{1}{\mu k}} \tag{25}$$

and for convexity:

$$\frac{\dfrac{\partial f(a)}{\partial a}}{f(a)} \leqslant \frac{\mu\left(V + \dfrac{a}{\rho}.V_0\right) - 1/k}{a.\mu.V_0 + V/k} \tag{26}$$

These conditions, especially the latter one, are often difficult to apply in this form, but in the transport-limited case with $f(a) \propto a^m$, they reduce to $m \geqslant 1$ for concavity, and $m \leqslant 0$ for convexity. At base level, there will be some concavity for all $m > 0$ (and also for all m if the base level is aggrading); and at the divide some convexity for all $m < 1$; so that for $0 < m < 1$ the profiles will always become convexo-concave. Convex divides and concave basal slopes can also arise, however, from more complex forms of the distance function, $f(a)$, for example, if $f(a)$ is equal to a constant plus a distance-increasing term, as might occur if combinations of creep and wash are acting.

Implicit in this kind of slope analysis is the concept that a characteristic-form slope profile might be analysed to give information about the processes producing it. To do this requires two sweeping assumptions, namely, that the profile has actually attained characteristic form and that the boundary conditions are as assumed. Only given these assumptions will the deduction of the process from the form be possible. It may be done on at least two levels of sophistication. The simplest level is to use equation (24) as an approximation to the form, and deduce the ratio m/n directly. At the next level of sophistication, transporting capacity can be derived directly from slope measurements by substitution in the differential equations, assuming μ, k, α are known.

If a series of profiles in a basin can be assumed to obey the same process law, then these derived values of transporting capacity can be correlated with slope and distance from divide. The necessary relationship is:

$$C \propto \frac{\mu \int_0^x V . \Omega . dx}{\Omega} + \frac{V}{k} \tag{27}$$

The practical difficulties of determining whether a given profile is a characteristic form without circular argument clearly limit this application, theoretically attractive though it is. Where it does seem reasonable to assume characteristic form, then the processes inferred must be those which actually formed the slope profile; these may contrast with present processes, for example in areas which formerly underwent rapid periglacial mass-wasting.

CONCLUSIONS

This paper is an attempt to examine some of the links between form and process, beginning with the continuity equation which is a general statement of conservation of mass, and linking it with empirical process laws. As many factors as possible have been left in the equations at each stage, to retain maximum flexibility in the solutions. By retaining contour curvature, for example, it is possible to consider both divides and river profiles as extreme examples of slope profiles, and thus to treat drainage basins as real geomorphological units. At many points, however, it has been convenient to make simplifying assumptions in order to obtain solutions, but the assumptions made are not the only ones which lead to reasonable solutions and the solutions in this paper should be considered only as examples of what can be done using a deterministic, rather than a probabilistic, model of landscape development.

To summarize, it is argued above that, given a fixed elevation for the base of a slope (or one varying according to an inverse power law), then empirical process laws can be used to calculate exact or approximate slope forms towards which hillslopes will develop as their initial profile form is gradually obliterated. General process conditions under which profiles tend to become convex or concave can be deduced. For example, if, on soil-covered slopes (transport-limited), the transport rate varies with distance from the divide as the mth power of the distance, then the profile will tend to become convex if $m \leqslant 0$, convexo-concave if $0 < m < 1$, and concave if $m \geqslant 1$. Of particular importance is the conclusion that river profiles in a humid environment will almost always tend to become concave.

In many cases the characteristic form towards which a profile tends will become indistinguishable from the actual profile at a time when the actual profile still has considerable relief, and this corresponds to a stage of Davisian maturity rather than old age. If such a characteristic form can be identified in the landscape, its form can be analysed to yield information about the processes which formed it, and their relative, though not absolute, rates.

These mathematical models are, therefore, an attempt to formalize process-response models of hillslopes into a single theory. On the one hand, process measurements can be used to predict the way in which hillslopes will develop, given the conditions of basal removal; on the other hand, slope profile measurements can sometimes be used to calculate the processes which formed them. This paper has concentrated on a particular type of interaction between these two approaches—that which leads to characteristic forms under constant base-level conditions. This type of solution is by no means the only, or even the most elegant, use of the continuity equation to analyse landscape development. For example it has been indicated above that soil development might readily be integrated into the model, and should be where chemical removal is important. The response of a landscape to sudden changes of climate or base level may be treated using kinematic wave theory to examine the propagation of knick-points through the landscape. Three-dimensional models in which contour curvatures may change and lines of greatest slope migrate laterally might lead to solutions which allow drainage densities to be predicted. These are only a few examples of the ways in which deterministic models based on the continuity equation and on process measurements can link process and form to show us how the landscape develops through cyclic time.

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RÉSUMÉ. *Modèles processus-réponse basé à l'équation de continuité.* L'équation différentielle de la continuité est la base d'une analyse théorique de la forme des versants. Si l'on considère la courbure des courbes de niveau en cette équation, les profils des versants et des fleuves sont comme une série continue, et l'on peut analyser ensemble ses membres. On déduit les profils des versants des fleuves avec quelque connaissance des processus de transport; et quelquefois on peut déduire les lois des processus des mesures de la forme. Si la ligne de partage et la surface de base ne bougent pas en sens horizontal, et puis le développement des versants mène à une « forme caractéristique » des versant et des fleuves, laquelle est indépendente de la forme initiale du profil, et dépend seulement des processus de transport solide. Le profil peut atteindre cette forme caractéristique pendant que 50–70 pour cent du relief initial y reste. À partir des études empiriques des processus, on montre que les formes caractéristiques sont convexes pour le « soil creep », concaves pour le « soil-wash » sans « gullies », et convexo-concaves pour « creep » et « wash » ensemble. Dans un bassin où l'on suppose que les profils des versants ont achevés ses formes caractéristiques, on peut analyser quelques mesures des profils pour rendre les lois des processus de transport, par lesquels se forme le bassin.

FIG. 1(a) Profil schématique d'un versant, montrant les axes x et y, et les symboles employés dans l'analyse théorique du développement des versants

(b) Carte schématique montrant les symboles employés dans l'analyse théorique du développement des versants où les lignes de gradient maximum convergent ou divergent (c'est à dire, où les courbes de niveau ne sont pas droites)

FIG. 2. Courbe sans dimensions montrant le développement d'un profile, initiallement droit, sous « soil creep » ($S = C = -\partial y/\partial x$). τ est une mesure sans dimensions du temps expiré.

FIG. 3. Courbe sans dimensions montrant la forme caractéristique ($V/V_0 = \cos \frac{1}{2}\pi(x/x_1)$) pour le développement d'un profil sous « soil creep » ($S = C = -\partial y/\partial x$); et les approximations obtenues comme bornes au-dessus et au-dessous de $V(x)$ en l'équation (20)

FIG. 4. Courbe sans dimensions de la solution approximative pour la forme caractéristique, montrant l'élevation V/V_0 comme une fonction de $I(x)/I(x_1)$, où

$$I(x) = \int_0^x \left\{\frac{\mu a + 1/k}{f(a)}\right\}^{1/n} . dx$$

(équation 21); et pour l'exponent $n = 1,0$ et $2,0$. On obtient chaque solution approximative comme le moyen des bornes en l'équation (20), avec λ choisi à satisfaire les conditions aux bornes à la surface de base ($V = 0$ à $x = x_1$).

FIG. 5. Courbe sans dimensions montrant les formes caractéristiques approximatives des profils pour les processus de Tableau I, pour le cas le plus simple $\mu = 1$, $k \to \infty$, $p \to \infty$. Les approximations sont à la même base qu'en la Figure 4.

ZUSAMMENFASSUNG. *Muster der Bergabhangprozessreaktion, auf der Stetigkeitsgleichung gegründet.* Die Stetigkeitsgleichung in ihrer Differentialform ist die Basis für eine theoretische Analyse der Abhangentwicklung. Indem man Begriffe der Höhenlinienkrümmung in der Gleichung einschliesst, kann man die Bergabhänge und Flussprofile als Teile einer ununterbrochenen Serie betrachten, und sie zusammen analysieren. Bergabhänge und Flussprofile können aus einer Kenntnis der Transportprozesse abgeleitet werden; und in einigen Fällen können Prozessregeln aus den Messungen der Formen gefolgert werden. Wenn die Wasserscheide und Erosionsbasis waagerecht festgelegt sind, führt die Abhangentwicklung zu einer ‚charakteristischen Form' der Bergabhänge und Flussprofile, welche von der ursprünglichen Form des Profils unabhängig ist, und nur von den Prozessen des Abfalltransport abhängt. Diese charakteristische Form mag erreicht werden, während 50–75 prozent des ursprünglichen Reliefs hinterbleibt. Wenn man bei empirischen Prozessen anfängt, zeigen sich die charakteristischen Formen als konvex für Erdgekriech, konkav für Erdspülung mit Wasserrinnen; und konvex-konkav für Erdspülung ohne Rinnen und für Zusammensetzungen der Gekriech und Spülung. Angenommen, dass in einem Becken die Abhangprofile ihre charakteristischen Formen erreicht haben, dort können Profilmessungen analysiert werden, um die Transportprozessregeln zu ergeben, durch die das Becken formiert wurde.

ABB. 1(a). Schematisches Abhangprofil, das die koordinierenden Achsen und Symbole zeigt, die in der theoretischen Analyse der Abhangentwicklung gebraucht werden

ABB. 1(b). Schematische Karte, die Symbole zeigt, welche in der theoretischen Analyse der Abhangentwicklung gebraucht werden, wo Linien des steilsten Hangs konvergieren oder divergieren (d.h. wo die Höhenschichtlinien nicht gerade sind)

ABB. 2. Dimensionslose graphische Darstellung, die die Abhangentwicklung eines anfänglich geraden Profils zeigt, unter dem Erdgekriech ($S = C = -\partial y/\partial x$). τ ist ein dimensionsloses Mass der verstrichenen Zeit.

ABB. 3. Graphische Darstellung ohne Dimension der charakteristischen Form ($V/V_0 = \cos \frac{1}{2}\pi(x/x_1)$) für das Abhangprofil, das sich unter dem Erdgekriech ($S = C = -\partial y/\partial x$); und die erhaltenen ungefähren Ergebnisse als die Ober- und Untergrenzen von $V(x)$ in der Gleichung (20)

ABB. 4. Graphische Darstellung ohne Dimension der annähernden charakteristischen Form Lösung fur Erhöhungsgrad V/V_0, in der Form von $(Ix)/I(x_1)$, wo die Werte

$$I(x) = \int_0^x \left\{\frac{\mu a + 1/k}{f(a)}\right\}^{1/n} .dx$$

(Gleichung 21); und für den Exponenten n die Werte 1,0, 2,0 nehmend. Die ungefähren Lösungen werden aus dem Mittel der Ober- und Untergrenzen in Gleichung (20) erhalten, mit λ gewählt, um die Grenzkondition an der Erosionsbasis zu genügen ($V = 0$ at $x = x_1$).

ABB. 5. Graphische Darstellung ohne Dimension der annähernden charakteristischen Form Hang Profile für eine Anzahl von Prozessen von Tabelle 1, für den einfachsten Fall $\mu = 1$, $k \to \infty$, $\rho \to \infty$. Annäherungen sind auf der gleichen Basis wie in Abb. 4.

Note: The symbol x_0 used on Figures 1 to 5 should be changed to x_1.